Modeling Bond Prices In Continuous-Time
Part III - Risky Bond Price, Duration and Convexity

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In this white paper we will build a model that calculates the market price of a risky (i.e. defaultable) coupon paying bond in continous-time. To that end we will use the following hypothetical problem from Part I...

Our Hypothetical Problem

The table below presents our go-forward model assumptions for the risk-free bond from Part I...

Table 1: Risk-Free Bond Assumptions

Symbol	Description	Balance
B	Bond face value	\$1,000
C	Annual coupon rate (%)	4.50
T	Term in years (#)	3.00

The table below presents our go-forward bond default assumptions...

Table 2: Bond Default Assumptions

Symbol	Description	Balance
R	Recovery rate given a bond default (%)	40.00
D	Cumulative default rate (%)	5.00
S	Credit spread over the risk-free rate (%)	2.00

We are tasked with answering the following questions:

Question 1: What is bond price given a continuous-time discount rate of 6.00% and 7.00%, respectively?

Question 2: Use duration and convexity to estimate the change in bond price in the question above.

Bond Price Equations

In Part I we defined the variable P_0 to be the price at time zero of a risk-free, coupon paying bond and the variable κ to be the continuous-time discount rate. Using the parameters in Table 1 above the equation for risk-free bond price at time zero is... [1]

$$P_0=\int\limits_0^T C\,B\,\mathrm{Exp}\left\{-\kappa\,u\right\}\delta u+B\,\mathrm{Exp}\left\{-\kappa\,T\right\} \tag{1}$$

Note that in Equation (1) above the bond is risk-free (i.e. cannot default) and therefore the bondholder is guaranteed to receive coupon payments over the time interval $[0,T]$ and face value at time T. If the bond is not risk-free and defaults at time t where $0\le t\le T$ then the bondholder receives coupon payments over the time interval $[0,t]$ and receives less than bond face value at time t.

To incorporate credit losses into the bond price equation above we will define the variable Γ_t to be the survival function at time t and the variable λ to be default intensity. The equation for the survival function is...

$$\Gamma_t = \mathrm{Exp}\left\{ -\lambda\, t \right\} \text{ ...where... } \lambda = \text{a value greater than zero} \tag{2}$$

Note that when t is zero then the survival function equals one and when t is greater than zero then the survival function is less than one but greater than zero. Using Equation (2) above note the following limit...

$$\lim_{t \to 0} \Gamma_t = \lim_{t \to 0} \mathrm{Exp}\left\{ -\lambda\, t \right\} = 0 \text{ ...when... } \lambda > 0 \tag{3}$$

A risky bond either survives or defaults over the time interval $[0, T]$. Note that the survival function starts out at one at time zero and then decreases over time due to periodic bond defaults. Using Equation (2) above the equation for the change in the survival function over time is...

$$\frac{\delta}{\delta t}\,\Gamma_t = -\lambda\,\mathrm{Exp}\left\{ -\lambda\, t \right\} \text{ ...such that... } \delta\,\Gamma_t = -\lambda\,\mathrm{Exp}\left\{ -\lambda\, t \right\} \delta t \tag{4}$$

We will define the cumulative default rate to be expected cumulative defaults over the time interval $[0, T]$ divided by bond face value. If we know the expected cumulative default rate at time T then by taking the natural log of both sides of Equation (2) above we can solve for the variable λ (default intensity) as follows...

$$\lambda = -\ln\left(\Gamma_T\right) \Big/ T = -\ln\left(1 - \text{cumulative default rate at time } T\right) \Big/ T \tag{5}$$

If the bond does not default over the time interval $[0, T]$ then the bondholder will receive bond value at time T. If this is the case then the expected face value received at time t is bond face value times the survival function at time T. Using Equations (1) and (2) above the equation for the present value of expected face value received at time T is...

$$\text{PV of } \mathbb{E}\left[\text{Face value received at time } T\right] = B\,\Gamma_T\,\mathrm{Exp}\left\{ -\kappa\, T \right\} \tag{6}$$

If the bond does default over the time interval $[0, T]$ then the bondholder receives coupon payments up until the time of default and zero coupon payments thereafter. The expected coupon payment received is therefore the contractual coupon payment times the survival function. Using Equations (1) and (2) above the equation for the present value of expected coupon payments received over the time interval $[0, T]$ is...

$$\text{PV of } \mathbb{E}\left[\text{Coupon payments received over the time interval } [0, T]\right] = \int_0^T C\,B\,\Gamma_t\,\mathrm{Exp}\left\{ -\kappa\, u \right\} \delta u \tag{7}$$

In Table 2 above we defined the variable R to be the credit loss recovery rate (i.e. percent of bond face value recovered given a bond default). If the bond does default over the time interval $[0, T]$ then the bondholder receives face value times the recovery rate at the time of default. Using Equations (1) and (2) above the equation for the present value of expected credit loss recoveries received over the time interval $[0, T]$ is...

$$\text{PV of } \mathbb{E}\left[\text{Credit loss recoveries received over the time interval } [0, T]\right] = \int_0^T \lambda\,R\,B\,\Gamma_t\,\mathrm{Exp}\left\{ -\kappa\, u \right\} \delta u \tag{8}$$

Using Equations (2), (6), (7) and (8) above the equation for risky bond price at time zero is...

$$\begin{aligned}
P_0 &= \int_0^T C\,B\,\Gamma_t\,\mathrm{Exp}\left\{ -\kappa\, u \right\} \delta u + \int_0^T \lambda\,R\,B\,\Gamma_t\,\mathrm{Exp}\left\{ -\kappa\, u \right\} \delta u + B\,\Gamma_T\,\mathrm{Exp}\left\{ -\kappa\, T \right\} \\
&= C\,B \int_0^T \mathrm{Exp}\left\{ -\lambda\, t \right\} \mathrm{Exp}\left\{ -\kappa\, u \right\} \delta u + \lambda\,R\,B \int_0^T \mathrm{Exp}\left\{ -\lambda\, t \right\} \mathrm{Exp}\left\{ -\kappa\, u \right\} \delta u + B\,\mathrm{Exp}\left\{ -\lambda\, T \right\} \mathrm{Exp}\left\{ -\kappa\, T \right\} \\
&= B\left(C + \lambda\,R\right) \int_0^T \mathrm{Exp}\left\{ -\lambda\, t \right\} \mathrm{Exp}\left\{ -\kappa\, u \right\} \delta u + B\,\mathrm{Exp}\left\{ -\lambda\, T \right\} \mathrm{Exp}\left\{ -\kappa\, T \right\} \\
&= B\left[\left(C + \lambda\,R\right) \int_0^T \mathrm{Exp}\left\{ -(\kappa + \lambda)\, u \right\} \delta u + \mathrm{Exp}\left\{ -(\kappa + \lambda)\, T \right\}\right]
\end{aligned} \tag{9}$$

Using Appendix Equation (27) below we can rewrite Equation (9) above is...

$$
\begin{aligned}
P_0 &= B\Bigg[\Bigg(C+\lambda\,R\Bigg)\Bigg(1-\text{Exp}\Bigg\{-(\kappa+\lambda)\,T\Bigg\}\Bigg)\Bigg(\kappa+\lambda\Bigg)^{-1}+\text{Exp}\Bigg\{-(\kappa+\lambda)\,T\Bigg\}\Bigg]\\
&= B\Bigg[\Bigg(C+\lambda\,R\Bigg)\Bigg(\kappa+\lambda\Bigg)^{-1}-\Bigg(C+\lambda\,R\Bigg)\text{Exp}\Bigg\{-(\kappa+\lambda)\,T\Bigg\}\Bigg(\kappa+\lambda\Bigg)^{-1}+\text{Exp}\Bigg\{-(\kappa+\lambda)\,T\Bigg\}\Bigg]\\
&= B\Bigg[\Bigg(C+\lambda\,R\Bigg)\Bigg(\kappa+\lambda\Bigg)^{-1}\Bigg(1-\text{Exp}\Bigg\{-(\kappa+\lambda)\,T\Bigg\}\Bigg)+\text{Exp}\Bigg\{-(\kappa+\lambda)\,T\Bigg\}\Bigg]
\end{aligned}
\tag{10}
$$

Using Equation (10) above the first derivative of bond price with respect to discount rate is...

$$
\frac{\delta}{\delta\kappa}\,P_0 = B\Bigg(\Bigg(C+\lambda\,R\Bigg)\frac{\delta}{\delta\kappa}\Bigg(\kappa+\lambda\Bigg)^{-1}-\Bigg(C+\lambda\,R\Bigg)\frac{\delta}{\delta\kappa}\,\text{Exp}\Bigg\{-(\kappa+\lambda)\,T\Bigg\}\Bigg(\kappa+\lambda\Bigg)^{-1}+\frac{\delta}{\delta\kappa}\,\text{Exp}\Bigg\{-(\kappa+\lambda)\,T\Bigg\}\Bigg)
\tag{11}
$$

Using Equation (10) above the second derivative of bond price with respect to discount rate is...

$$
\frac{\delta^2}{\delta\kappa^2}\,P_0 = B\Bigg(\Bigg(C+\lambda\,R\Bigg)\frac{\delta^2}{\delta\kappa^2}\Bigg(\kappa+\lambda\Bigg)^{-1}-\Bigg(C+\lambda\,R\Bigg)\frac{\delta^2}{\delta\kappa^2}\,\text{Exp}\Bigg\{-(\kappa+\lambda)\,T\Bigg\}\Bigg(\kappa+\lambda\Bigg)^{-1}+\frac{\delta^2}{\delta\kappa^2}\,\text{Exp}\Bigg\{-(\kappa+\lambda)\,T\Bigg\}\Bigg)
\tag{12}
$$

Change in Bond Price

Using Equations (11) and (12) above we will define duration and convexity to be the following equations...

$$
\text{Duration} = \frac{\delta}{\delta\kappa}\,P_0\Big/B \quad \text{....and...} \quad \text{Convexity} = \frac{\delta^2}{\delta\kappa^2}\,P_0\Big/B
\tag{13}
$$

Using Equation (13) above the equation for the change in bond price with respect to a change in discount rate via a Taylor Series Expansion of order two is...

$$
\delta\,P_0 = \Bigg(B\times\text{Duration}\times\delta\kappa\Bigg)+\Bigg(B\times\frac{1}{2}\times\text{Convexity}\times\delta\kappa^2\Bigg)
\tag{14}
$$

Using Appendix Equation (32) below the solution to the duration equation in Equations (13) and (14) above is...

$$
\text{Duration} = -\Bigg(C+\lambda\,R\Bigg)\Bigg(1-\text{Exp}\Bigg\{-(\kappa+\lambda)\,T\Bigg\}\Bigg(1+(\kappa+\lambda)\,T\Bigg)\Bigg)\Bigg(\kappa+\lambda\Bigg)^{-2}-T\,\text{Exp}\Bigg\{-(\kappa+\lambda)\,T\Bigg\}
\tag{15}
$$

Using Appendix Equation (33) below the solution to the convexity equation in Equations (13) and (14) above is...

$$
\text{Convexity} = \Bigg(C+\lambda\,R\Bigg)\Bigg(2-\text{Exp}\Bigg\{-(\kappa+\lambda)\,T\Bigg\}\Bigg((\kappa+\lambda)^2T^2+2\,(\kappa+\lambda)\,T+2\Bigg)\Bigg(\kappa+\lambda\Bigg)^{-3}+T^2\,\text{Exp}\Bigg\{-(\kappa+\lambda)\,T\Bigg\}
\tag{16}
$$

The Answers To Our Hypothetical Problem

Question 1: What is bond price given a continuous-time discount rate of 6.00% and 7.00%, respectively?

First part of question:

Using the parameters in Tables 1 and 2 above the risky bond discount rate is...

$$
\kappa = \text{Risk-free bond rate} + \text{Credit spread} = 0.0600 + 0.0200 = 0.0800
\tag{17}
$$

Using the parameters in Tables 1 and 2 above and Equation (5) above the equation for default intensity is...

$$
\lambda = -\ln\Bigg(1-0.05\Bigg)\Big/3.00 = 0.0171
\tag{18}
$$

Using Equation (10) above the answer to our problem is...

$$P_0 = 1,000 \times \Bigg[\bigg(0.045 + 0.0171 \times 0.40\bigg)\bigg(0.0800 + 0.0171\bigg)^{-1}\bigg(1 - \text{Exp}\bigg\{-(0.0800 + 0.0171) \times 3\bigg\}\bigg) + \text{Exp}\bigg\{-(0.0800 + 0.0171) \times 3\bigg\}\Bigg] = 882.21 \tag{19}$$

Second part of question:

Using the parameters in Tables 1 and 2 above the risky bond discount rate is...

$$\kappa = \text{Risk-free bond rate} + \text{Credit spread} = 0.0700 + 0.0200 = 0.0900 \tag{20}$$

Using Equation (10) above the answer to our problem is...

$$P_0 = 1,000 \times \Bigg[\bigg(0.045 + 0.0171 \times 0.40\bigg)\bigg(0.0900 + 0.0171\bigg)^{-1}\bigg(1 - \text{Exp}\bigg\{-(0.0900 + 0.0171) \times 3\bigg\}\bigg) + \text{Exp}\bigg\{-(0.0900 + 0.0171) \times 3\bigg\}\Bigg] = 858.22 \tag{21}$$

Question 2: Use duration and convexity to estimate the change in bond price in the question above.

$$\text{Model Parameters: } \kappa = 0.0800 \text{ ...and... } \delta\kappa = 0.0900 - 0.0800 = 0.0100 \tag{22}$$

Using the model parameters in Equation (22) above and duration Equation (15) above the equation for duration for our problem is...

$$\text{Duration} = -\bigg(0.045 + 0.0171 \times 0.40\bigg) \times \bigg(1 - \text{Exp}\bigg\{-(0.0800 + 0.0171) \times 3\bigg\} \times \bigg(1 + (0.0800 + 0.0171) \times 3\bigg)\bigg) \times \bigg(0.0800 + 0.0171\bigg)^{-2} - 3 \times \text{Exp}\bigg\{-(0.0800 + 0.0171) \times 3\bigg\} = -2.4345 \tag{23}$$

Using the model parameters in Equation (22) above and convexity Equation (16) above the equation for convexity for our problem is...

$$\text{Convexity} = \bigg(0.045 + 0.0171 \times 0.40\bigg) \times \bigg(2 - \text{Exp}\bigg\{-(0.0800 + 0.0171) \times 3\bigg\} \times \bigg((0.0800 + 0.0171)^2 \times 3^2 + 2 \times (0.0800 + 0.0171) \times 3 + 2\bigg) \times \bigg(0.0800 + 0.0171\bigg)^{-3} + 3^2 \times \text{Exp}\bigg\{-(0.0800 + 0.0171) \times 3\bigg\} = 7.1013 \tag{24}$$

Using Equations (14), (23) and (24) above the answer to our problem is...

$$\text{Change in Bond Price} = 1,000 \times -2.4345 \times 0.01 + 1,000 \times \frac{1}{2} \times 7.1013 \times 0.01^2 = -24.00 \tag{25}$$

Note that using Equations (17) and (18) above the actual change in bond price was...

$$\text{Change in Bond Price} = 882.21 - 858.22 = -23.99 \tag{26}$$

Appendix

A. The solution to the following integral is...

$$\int_0^T \text{Exp}\bigg\{-(\kappa + \lambda)\,u\bigg\}\,\delta u = -(\kappa + \lambda)^{-1}\,\text{Exp}\bigg\{-(\kappa + \lambda)\,u\bigg\}\bigg[_0^T = (\kappa + \lambda)^{-1}\bigg(1 - \text{Exp}\bigg\{-(\kappa + \lambda)\,T\bigg\}\bigg) \tag{27}$$

B. The first and second derivative of the following equation is...

$$\frac{\delta}{\delta\kappa}\left(\kappa+\lambda\right)^{-1}=-\left(\kappa+\lambda\right)^{-2}\text{ ...and... }\frac{\delta^2}{\delta\kappa^2}\left(\kappa+\lambda\right)^{-1}=2\left(\kappa+\lambda\right)^{-3}\tag{28}$$

C. The first and second derivative of the following equation is...

$$\frac{\delta}{\delta\kappa}\,\mathrm{Exp}\left\{-\left(\kappa+\lambda\right)u\right\}=-u\,\mathrm{Exp}\left\{-\left(\kappa+\lambda\right)u\right\}\text{ ...and... }\frac{\delta^2}{\delta\kappa^2}\,\mathrm{Exp}\left\{-\left(\kappa+\lambda\right)u\right\}=u^2\,\mathrm{Exp}\left\{-\left(\kappa+\lambda\right)u\right\}\tag{29}$$

D. Using Appendix Equation (28) and (29) above the first derivative of the following equation via the product rule is...

$$\frac{\delta}{\delta\kappa}\left(\kappa+\lambda\right)^{-1}\mathrm{Exp}\left\{-\left(\kappa+\lambda\right)u\right\}=-\left(\kappa+\lambda\right)^{-2}\mathrm{Exp}\left\{-\left(\kappa+\lambda\right)u\right\}\left(1+\left(\kappa+\lambda\right)u\right)\tag{30}$$

The second derivative of Equation (30) above is...

$$\frac{\delta^2}{\delta\kappa^2}\left(\kappa+\lambda\right)^{-1}\mathrm{Exp}\left\{-\left(\kappa+\lambda\right)u\right\}=\left(\kappa+\lambda\right)^{-3}\mathrm{Exp}\left\{-\left(\kappa+\lambda\right)u\right\}\left((\kappa+\lambda)^2u^2+2\left(\kappa+\lambda\right)u+2\right)\tag{31}$$

E. Using Equation (11) above and Appendix Equations (28), (29) and (30) above the solution to the duration equation in Equations (13) and (14) above is...

$$\begin{aligned}\text{Duration}&=\frac{\delta}{\delta\kappa}\,P_0\Big/B\\&=-\left(C+\lambda R\right)\left(\kappa+\lambda\right)^{-2}+\left(C+\lambda R\right)\mathrm{Exp}\left\{-\left(\kappa+\lambda\right)T\right\}\left(1+\left(\kappa+\lambda\right)T\right)\left(\kappa+\lambda\right)^{-2}-T\,\mathrm{Exp}\left\{-\left(\kappa+\lambda\right)T\right\}\\&=-\left(C+\lambda R\right)\left(1+\mathrm{Exp}\left\{-\left(\kappa+\lambda\right)T\right\}\left(1+\left(\kappa+\lambda\right)T\right)\right)\left(\kappa+\lambda\right)^{-2}-T\,\mathrm{Exp}\left\{-\left(\kappa+\lambda\right)T\right\}\end{aligned}\tag{32}$$

F. Using Equation (12) above and Appendix Equations (28), (29) and (31) below the solution to the convexity equation in Equations (13) and (14) above is...

$$\begin{aligned}\text{Convexity}&=\frac{\delta^2}{\delta\kappa^2}\,P_0\Big/B\\&=2\left(C+\lambda R\right)\left(\kappa+\lambda\right)^{-3}-\left(C+\lambda R\right)\mathrm{Exp}\left\{-\left(\kappa+\lambda\right)T\right\}\left((\kappa+\lambda)^2T^2+2\left(\kappa+\lambda\right)T+2\right)\left(\kappa+\lambda\right)^{-3}\\&\quad+T^2\,\mathrm{Exp}\left\{-\left(\kappa+\lambda\right)T\right\}\\&=\left(C+\lambda R\right)\left(2-\mathrm{Exp}\left\{-\left(\kappa+\lambda\right)T\right\}\left((\kappa+\lambda)^2T^2+2\left(\kappa+\lambda\right)T+2\right)\left(\kappa+\lambda\right)^{-3}+T^2\,\mathrm{Exp}\left\{-\left(\kappa+\lambda\right)T\right\}\right.\end{aligned}\tag{33}$$

References

[1] Gary Schurman, *Modeling Bond Price in Continuous-Time - Part I,* November, 2020.